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A case study of methods of series summation: Kelvin–Helmholtz instability of finite amplitude

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Abstract

We compute the singularities of the solution of the Birkhoff–Rott equation that governs the evolution of a planar periodic vortex sheet. Our approach uses the Taylor series obtained by Meiron et al. [J. Fluid Mech. 114 (1982) 283] for a flat sheet subject initially to a sinusoidal disturbance of amplitude *a*. The series is then summed by using various generalisations of the Padé method. We find approximate values for the location and type of the principal singularity as *a* ranges from zero to infinity. Finally, the results are used as a basis to guide the choice of methods of summing series arising from problems in fluid mechanics.

Keywords: Kelvin–Helmholtz instability; Birkhoff–Rott equation; Series summation; Hermite–Padé approximation

1. Introduction

Many believe that diverse solutions of mathematical problems of fluid mechanics with smooth initial data develop singularities after finite times. The question of existence and regularity of solutions for large times is fundamental not only for its mathematical importance but also for its implications about the physical modelling of real flows by the Euler and Navier–Stokes equations. Indeed, this issue is at the heart of one of the prize millenium problems set by the Clay Institute [3].

A common method to solve such initial-value problems has been to expand the solution in powers of the time, compute several terms of the power series, and sum the series by rational Padé approximation or its generalisations [8,10,12]. This method cannot *prove* that a singularity does develop after a finite time, but often produces convincing approximations of the time when a singularity appears to develop. These approximations are widely accepted by the fluid mechanics community.

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This paper focuses on one such initial-value problem, Kelvin–Helmholtz instability of finite amplitude, using it as a case study to illustrate the merits and demerits of various methods of series summation. As we shall see, in this particular case, the comparison suggests that rational Padé approximation, the most widely used method of series summation, is inferior to other methods. We offer some general guidance to choose appropriate methods for other types of problems.

Kelvin–Helmholtz instability concerns the growth of perturbations of the plane interface between two uniform parallel streams of incompressible inviscid fluids, and in particular the perturbation of a vortex sheet in one fluid. We shall confine our analysis to the special case of a periodic vortex sheet in one fluid. Then the evolution of the perturbed vortex sheet is governed by the periodic form of the Birkhoff–Rott equation [2,10]

$$\frac{\partial z^*}{\partial t}(e,t) = \frac{1}{4\pi \mathrm{i}}\,\mathrm{P.V.}\int_0^{2\pi} \gamma(e')\cot\left(\frac{z(e,t)-z(e',t)}{2}\right)\mathrm{d}e'. \tag{1}$$

The integral in this expression is a Cauchy principal value integral, e is a Lagrangian marker variable, t represents time, $z = x + \mathrm{i}y$ is a parametric representation of the sheet, the asterisk denotes complex conjugation and γ is related to the vortex sheet strength – essentially, the jump across the sheet in the tangential fluid velocity.

In his seminal paper, Moore [11] studied the particular case

$$z(e,0) = e + \varepsilon \sin e, \quad \gamma(e) = 1,\ \varepsilon > 0. \tag{2}$$

By considering the initial-value problem (1) + (2) in Fourier space, Moore found a relatively simple approximation of the Birkhoff–Rott equation, and showed that the solution of this simplified model has a real singularity that corresponds to a point of infinite curvature on the vortex sheet. This led him to conjecture that the solution of the full model (1) + (2) also has a real singularity at some critical time, say t_c. This conjecture has not yet been proved, but there is much compelling evidence to support it.

Siegel [15] and Cowley et al. [4] present systematic attempts to improve Moore's simplified model by using analytic continuation in the variable e. The equations obtained in this way are asymptotic approximations of the Birkhoff–Rott equations in the limit as $\mathrm{Im}\, e \to \infty$ and, therefore, cannot be expected to yield quantitative information on the location of the real singularities. Nevertheless, the qualitative inference from these studies is that complex singularities of the vortex sheet move in the e plane until one reaches the real axis at the critical time t_c. Krasny [9], Shelley [14] and others provided further support for the existence of a critical time by direct numerical simulation of the Birkhoff–Rott equation. Typically, such simulations are accurate as long as the solution remains smooth. The erratic, mesh-dependent behaviour of the numerical solution for large times is symptomatic of the existence of a critical time.

If a real singularity exists, then it must be possible to compute it to any desired accuracy by performing enough arithmetic operations. The best strategy for locating the singularity of an analytic function accurately is to process its Taylor series coefficients [18]. This is the approach taken by Meiron et al. [10], who consider the particular case where the sheet is initially flat and set in motion impulsively by a sinusoidal disturbance of γ:

$$z(e,0) = e, \quad \gamma(e) = 1 + a\cos e,\ a = \text{const}. \tag{3}$$

They expand the solution in a Taylor series in powers of t and, using Padé approximants, compute t_c for various values of a with a relative error that is roughly 2%. One purpose of this paper is to re-examine the Taylor series for the solution of the problem (1) + (3) and, by use of powerful generalisations of the Padé method, to compute the critical time t_c for a wide range of the parameter a to an accuracy much greater than that achieved by Meiron et al. [10]. As a practical application, we study the dependence of the critical time upon the parameter a. Other calculations are performed: We compute, for a fixed a, the motion of the singularity in the complex e plane as time evolves. We also compute the first two terms in the expansion of the solution in fractional powers of $t_c - t$ at the singularity.

The remainder of the paper is as follows: In Section 2, we derive a recurrence relation for the Taylor coefficients of the solution z of the problem (1) + (3). Our approach is very similar to that of Meiron et al. [10], but our treatment of the troublesome principal value integral is more efficient. With the benefit of twenty years of advances in computing hardware, we are able to find many more terms of the series for general a in exact arithmetic. The methods used to sum the series are described in Section 3. Our main tool is the particular type of algebraic approximant developed by Drazin and Tourigny [5,6]. In addition, we shall use differential approximants to compute the exponent of the singularity. Section 4 is devoted to a detailed study of the particular case $a = 1$. The limit as $a \to \infty$ is studied in Section 5. Finally, we present our conclusions in Section 6.

2. A Taylor series

When $a = 0$, the initial-value problem (1) + (3) is easily solved. It is therefore natural to seek an expansion in ascending powers of a. By direct calculation, it is found that

$$z(e,t) = e + \sum_{n=1}^{\infty} a^n \left(\sum_{k=1}^{n} z_n^{(k)}(t) \sin(ke) \right) \quad \text{as } a \to 0. \tag{4}$$

Hence the coefficient of each power of a is a finite sum of sines. The coefficients $z_n^{(k)}$ are made up of powers of t and of exponentials of t. (Explicit formulae for expressions that are essentially equivalent to the first few $z_n^{(k)}$ are given in [10, p. 287].) The upshot of Moore's analysis is that, in the limit as $t \to \infty$, the nth coefficient is asymptotic to

$$\frac{\mathrm{i}-1}{2} \frac{n^{n-2}}{n!} \left(\frac{t}{4}\right)^{n-1} \exp\left(n\frac{t}{2}\right) \sin(ne).$$

It is tempting to deduce from this observation that

$$z(e,t) \sim e + \frac{2\mathrm{i}-2}{t} \sum_{n=1}^{\infty} \left[nt\frac{a}{4} \exp\left(\frac{t}{2}\right) \right]^n \frac{\sin(ne)}{n^2 n!} \quad \text{as } a \to 0. \tag{5}$$

A rigorous justification of this relationship would be a major undertaking, for there are various competing limits involved, namely $a \to 0$, $n \to \infty$ and $t \to \infty$. Nevertheless, if we proceed heuristically and accept the validity of the formula (5), we see that the Fourier coefficients of z decay exponentially as $n \to \infty$ up to some critical time t_c given asymptotically by the formula

$$\ln t_c + \frac{t_c}{2} + 1 \sim \ln\frac{4}{a} \quad \text{as } a \to 0. \tag{6}$$

The dashed curve in Fig. 1 is obtained by using this formula.

Although quite a few of the $z_n^{(k)}$ can be found explicitly by use of symbolic algebra programmes such as MAPLE, the expressions obtained are so complicated that the series (4) is unsuitable for numerical work. We shall therefore imitate Meiron et al. [10] and compute the terms in the Taylor series

$$z(e,t) = \sum_{n=0}^{\infty} Z_n(e) t^n \quad \text{as } t \to 0. \tag{7}$$

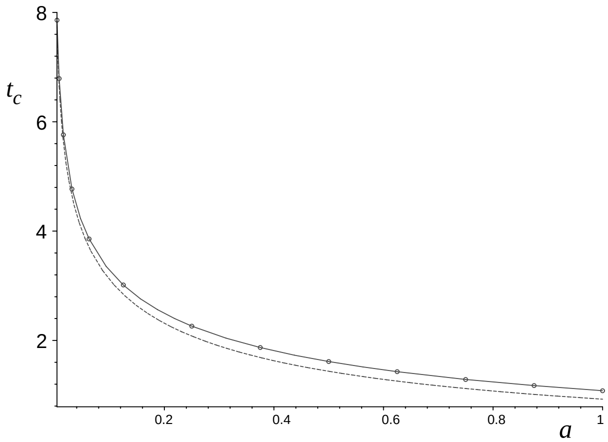

Fig. 1. The critical time t_c as a function of a. The dashed curve shows Moore's approximation given by the asymptotic formula (6). The solid curve is computed by the methods of Section 3.

Clearly, in view of the form of the coefficients in the expansion (4), we can write

$$Z_n(e) = \sum_{k=1}^{n} Z_n^{(k)} \sin(ke) \quad \text{for } n \geqslant 1, \tag{8}$$

where the $Z_n^{(k)}$ are polynomials in a.

In order to find a recurrence relation for the Z_n, we introduce a series for the cot term

$$\cot\left(\frac{z(e,t) - z(e',t)}{2}\right) =: \sum_{n=0}^{\infty} Y_n(e,e')t^n.$$

The Birkhoff–Rott equation (1) then gives

$$(n+1)Z_{n+1}^*(e) = \frac{1}{4\pi \mathrm{i}} \mathrm{P.V.} \int_0^{2\pi} (1 + a\cos e')Y_n(e,e')\,\mathrm{d}e'. \tag{9}$$

From the identity

$$\frac{\mathrm{d}}{\mathrm{d}u} \cot u = -1 - \cot^2 u,$$

we deduce the following recurrence relation for Y_n:

$$nY_n = -W_n - \sum_{k=1}^{n} W_k V_{n+1-k}, \tag{10}$$

where

$$W_n(e,e') = \frac{n}{2}[Z_n(e) - Z_n(e')] \text{ and } V_n = \sum_{k=0}^{n} Y_k Y_{n-k}, \quad n \in \mathbb{N}. \tag{11}$$

The main difficulty is the evaluation of the Cauchy principal value integral on the right-hand side of Eq. (9). We could follow Meiron et al. [10] and regularise the integrand by subtracting from it the product

$$(1 + a\cos e)\cot\left(\frac{z(e,t) - z(e',t)}{2}\right)\frac{\partial z(e',t)/\partial e'}{\partial z(e,t)/\partial e},$$

whose principal value integral vanishes. The integral would become a proper one, but at the price of introducing an additional multiplication and division. We choose to take a different route. First, introduce the new variable

$$\theta = \frac{e - e'}{2},$$

so that (9) becomes

$$(n+1)Z^*_{n+1}(e) = \frac{1}{2\pi \mathrm{i}}\,\mathrm{P.V.}\int_{e/2-\pi}^{e/2} [1 + a\cos e\cos(2\theta) + a\sin e\sin(2\theta)]Y_n(e, e-2\theta)\,\mathrm{d}\theta. \tag{12}$$

For the first few values of n, direct calculations reveal that

$$\sin\theta Y_n(e, e-2\theta) = Y_n^{(1)}(\theta) + \sum_{k=1}^{n}\left[\sin\theta Y_n^{(2k)}(\theta)\sin(ke) + Y_n^{(2k+1)}(\theta)\cos(ke)\right],$$

where $Y_n^{(k)}(\theta)$ is in fact a polynomial in $\cos\theta$ having the same parity as k. The integrand on the right-hand side of (12) may be expanded as a trigonometric polynomial in e. Noting that $Z_n(e)$ is a finite sum of sines, we may drop the terms in the integrand that result in cosines to obtain

$$\begin{aligned}(n+1)Z^*_{n+1}(e) = \frac{1}{2\pi \mathrm{i}}\,\mathrm{P.V.}\int_{e/2-\pi}^{e/2} \Bigg\{ & 2a\cos\theta Y_n^{(1)}(\theta)\sin e + [1 + a\cos(2\theta)\cos e]\sum_{k=1}^{n} Y_n^{(2k)}(\theta)\sin(ke) \\ & + 2a\cos\theta\sin e\sum_{k=1}^{n} Y_n^{(2k+1)}(\theta)\cos(ke)\Bigg\}\,\mathrm{d}\theta.\end{aligned}$$

The integrand on the right-hand side of this last equation is an even polynomial in $\cos\theta$ and, hence, a perfectly regular function of θ with period π. The integral is proper and we may thus write

$$\begin{aligned}(n+1)Z^*_{n+1}(e) = \frac{1}{2\pi \mathrm{i}}\int_{0}^{\pi} \Bigg\{ & 2a\cos\theta Y_n^{(1)}(\theta)\sin e + [1 + a\cos(2\theta)\cos e]\sum_{k=1}^{n} Y_n^{(2k)}(\theta)\sin(ke) \\ & + 2a\cos\theta\sin e\sum_{k=1}^{n} Y_n^{(2k+1)}(\theta)\cos(ke)\Bigg\}\,\mathrm{d}\theta,\end{aligned} \tag{13}$$

which is easily evaluated.

In summary, using

$$Z_0(e) = e \quad \text{and} \quad Y_0(e, e') = \cot\left(\frac{e - e'}{2}\right),$$

we can compute Z_n for $n \geqslant 1$ by recurrence via the relations (10), (11) and (13). Although the computational complexity increases rapidly, we managed to compute Z_n in exact arithmetic for $0 \leqslant n \leqslant 43$ and general a. The computational work is much reduced if one computes the coefficients for specific values of a. Thus, for $a = 1$ and $a = \infty$ (see Sections 4 and 5), we computed the first 60 and 77 coefficients, respectively.

3. Hermite–Paé approximants

Meiron et al. [10] use Padé approximants in order to compute the critical time t_c for various values of a. In this section, we describe a wide class of approximants – including Padé – and discuss some of the principles that should guide their use. For the sake of brevity, we have confined our discussion to the class of *Hermite–Padé* approximants since these are arguably the most widely used in the applications. However, it should be noted that there exist other useful approximants outside this class (see [1]), and other useful methods of series summation that do not rely on approximants (see [19]).

We say that a function is an *approximant* for the series

$$U := \sum_{n=0}^{\infty} u_n \lambda^n \tag{14}$$

if it shares with U the same first few Taylor coefficients at $\lambda = 0$. Thus, the simplest approximants are the partial sums of the series U. When the series converges rapidly, such *polynomial* approximants can provide good approximations of the sum. In practice, however, the presence of singularities in the complex λ plane often prevents rapid convergence. It is then necessary to seek approximants in a larger class of functions.

In the Padé method, the approximant is sought in the class of *rational* functions. When applied to the Nth partial sum of the series (14), it involves the construction of two polynomials in λ, say $P_N^{(0)}$ and $P_N^{(1)}$, such that

$$\deg P_N^{(0)} + \deg P_N^{(1)} + 1 = N \quad \text{and} \quad P_N^{(0)} U + P_N^{(1)} = \mathrm{O}(\lambda^N) \quad \text{as } \lambda \to 0. \tag{15}$$

The rational approximant U_N is then defined by

$$P_N^{(0)} U_N + P_N^{(1)} = 0. \tag{16}$$

We emphasise that only the first N coefficients u_n of the series U are required in order to construct the $P_N^{(\ell)}$. The second equation in (15) can be expressed as a linear system of equations for the unknown coefficients of the polynomials $P_N^{(\ell)}$. In order to obtain a unique solution, one must normalise in some way; for example by setting

$$P_N^{(0)}(0) = 1.$$

The first equation in (15) then simply ensures that the matrix associated with the system is square. The poles of U_N are the zeroes of the polynomial $P_N^{(0)}$ and the hope is that one of these zeroes, say $\lambda_{c,N}$, tends to the location, say λ_c, of the dominant singularity of U as N increases. In practice, one finds that the method is most accurate when the dominant singularity of U is a pole.

The principle underlying any approximant method is that the (difficult) problem to which it is applied (e.g. the Birkhoff–Rott equation) is replaced by a more tractable problem (e.g. Eq. (16)) involving polynomial coefficients. Given a particular problem, the ideal approximant method replaces it by one that is rich enough to reproduce the essential features of the true solution, but simple enough that these features can be deduced easily once the polynomial coefficients are known.

Hermite–Padé approximation generalises the Padé method in the following sense: Given d power series $U^{(1)}, \ldots, U^{(d)}$, one constructs polynomials $P_N^{(\ell)}$ such that

$$\deg P_N^{(0)} + \deg P_N^{(1)} + \cdots + \deg P_N^{(d)} + d = N \tag{17}$$

and

$$P_N^{(0)} U^{(d)} + P_N^{(1)} U^{(d-1)} + \cdots + P_N^{(d)} = \mathrm{O}(\lambda^N) \quad \text{as } \lambda \to 0. \tag{18}$$

For functions with logarithmic or algebraic singularities, special types of Hermite–Padé approximants can sometimes outperform the Padé method. For instance, suppose that

$$U(\lambda) - U_0 \sim U_1\,(\lambda_c - \lambda)^{\alpha} \quad \text{as } \lambda \to \lambda_c. \tag{19}$$

Then a good summation method is to choose a suitably large natural number d, set

$$U^{(\ell)} = U^{\ell}, \quad 1 \leqslant \ell \leqslant d,$$

and define an *algebraic* approximant U_N of U by

$$P_N^{(0)}\,U_N^{d} + P_N^{(1)}\,U_N^{d-1} + \cdots + P_N^{(d)} = 0. \tag{20}$$

Note that, for $d > 1$, U_N is a multivalued function of λ with d branches. Furthermore, its singularities are of the form

$$U_N(\lambda) - U_{0,N} \sim U_{1,N}\,(\lambda_{c,N} - \lambda)^{\alpha_N} \quad \text{as } \lambda \to \lambda_{c,N}, \tag{21}$$

where the numbers $U_{0,N}$, $U_{1,N}$, α_N and $\lambda_{c,N}$ can be deduced easily from the polynomials $P_N^{(\ell)}$. The precise formulae are given in the appendix. Thus, if the assumption (19) is valid, then these numbers can provide good approximations of the true singularity parameters U_0, U_1, α and λ_c for N sufficiently large.

Alternatively, one may use

$$U^{(\ell)} = D^{\ell-1}U, \quad 1 \leqslant \ell \leqslant d,$$

where D denotes differentiation with respect to λ, and define a (single-valued) *differential* approximant U_N of U by

$$P_N^{(0)}\,D^{d-1}U_N + P_N^{(1)}D^{d-2}U_N + \cdots + P_N^{(d)} = 0. \tag{22}$$

The singularities of U_N need not always be of the form (21) but, when they are, the numbers α_N and $\lambda_{c,N}$ can again be obtained by formulae analogous to the algebraic case; these can be found in the appendix.

In order to characterise the approximant U_N completely, one must choose the degree of each of the polynomials $P_N^{(\ell)}$ in (17). A popular strategy is to take

$$\deg P_N^{(0)} = \cdots = \deg P_N^{(d)}, \quad d\text{fixed and } N \to \infty. \tag{23}$$

Sergeyev [13] showed how such *diagonal* approximants can be constructed by recurrence: First, choose polynomials $P_N^{(\ell)}$ such that (18) holds for $N = 0, \ldots, d$. Next, given $P_N^{(\ell)}$ for $N = n-1, \ldots, n+d-1$, set

$$P_{n+d}^{(\ell)} = \lambda P_{n-1}^{(\ell)} + \sum_{k=1}^{d} c_n^{(\ell)} P_{n+k-1}^{(\ell)}, \tag{24}$$

where the coefficients $c_n^{(\ell)}$ are chosen so that (18) holds for $N = n + d$.

One must also choose the starting polynomials. In the case of algebraic approximants, we follow Sergeyev [13] in taking $P_N^{(\ell)}$ so that

$$P_N^{(0)}U^d + P_N^{(1)}U^{d-1} + \cdots + P_N^{(d)} = [U(\lambda) - U(0)]^N \quad \text{for } N = 0, 1, \ldots, d.$$

In the case of differential approximants, we set, for $N = 0, 1, \ldots, d$,

$$P_N^{(\ell)} = \begin{cases} 0 & \text{if } \ell < d - N, \\ 1 & \text{if } \ell = d - N, \end{cases}$$

and let $P_N^{(\ell)}$ for $\ell > d - N$ be constants such that (18) holds. Such constants can be calculated by solving a simple linear system of equations.

Drazin and Tourigny [5] proposed a different strategy for constructing algebraic approximants in which one takes

$$\deg P_N^{(\ell)} = \ell \text{ for } 0 \leqslant \ell \leqslant d \quad \text{and} \quad N = \frac{d}{2}(d+3) \text{ as } d \to \infty. \tag{25}$$

This has been shown to be a good approach when the analytic function represented by the series U has a countable infinity of branches [17].

4. Disturbances of unit initial amplitude

There is a general consensus that, for a suitable choice of functional U, the model (19) with $\lambda = t$ is an adequate description of the dominant singularity that develops on the vortex sheet. In order to identify the particular type of Hermite–Padé approximant that is best for the computation of t_c and α, we now consider the particular case $a = 1$ in some depth. We emphasise that there is nothing really special about this choice of a; we expect our findings to be typical of the general case where a is finite and non-zero.

The coefficients Z_n of the Taylor series (7) are trigonometric polynomials in e, and it is impractical to calculate Hermite–Padé approximants with such complicated expressions. For this reason, Meiron et al. [10] applied the Padé method to the series for

$$\int_0^{2\pi} \left| \frac{\partial^p z}{\partial e^p}(e,t) \right|^{2p} \mathrm{d}e, \quad p = 1, 2, 4. \tag{26}$$

This functional is an even function of t and so has the disadvantage that the number of coefficients available for the analysis is effectively halved – an undesirable loss of information. We prefer to use

$$U(\lambda) := \frac{\partial z}{\partial e}(0,t), \quad \lambda = t. \tag{27}$$

In Tables 1 and 2, we show the estimates of t_c obtained by using various differential or algebraic approximants. These estimates, which we denote by $t_{c,N}$, are calculated by using the formulae for $\lambda_{c,N}$ in Appendix A. Since we expect the critical time to be real, the imaginary part of $t_{c,N}$ gives a lower bound for the error and thus some measure of the accuracy attained. The results indicate that better approximations are obtained by taking $d > 1$. There might be a very slight advantage in using the method of Drazin and Tourigny [5] rather than the usual strategy of keeping d fixed as N increases (see the third column of Table 3). We conclude with reasonable confidence that

$$t_c = 1.080316 \pm 10^{-6}.$$

Cowley et al. [4] analysed the local form of the singularity and obtained the asymptotic relation

$$\frac{\partial z}{\partial e}(0,t) - U_0 \sim U_1(t_c - t)^{\alpha} \quad \text{as } t \to t_c \tag{28}$$

with $\alpha = 1/2$. We can use approximants to determine α, U_0 and U_1 directly. First, from the formula (50) with $\lambda = t$, we obtain estimates for α by using differential approximants. The results shown in Table 2 suggest that

$$\alpha = \frac{1}{2} \pm 10^{-2}.$$

Table 1
Estimate $t_{c,N}$ of t_c for the series (27) obtained by diagonal algebraic approximants as N increases

N	$d=1$	$d=2$	$d=3$
5	2.0761873 − 0.7941118i	0.9011885 + 0.1489890i	0.8999586 + 0.1453168i
9	1.3587025 − 0.0337188i	1.0913431 − 0.0073643i	1.1162063 − 0.0346961i
14	1.2144946 − 0.0011937i	1.0797230 − 0.0026782i	1.0803420 − 0.0033167i
20	1.1488424 − 0.0174031i	1.0802010 − 0.0009084i	1.0802406 − 0.0005642i
27	1.1075638 + 0.0045061i	1.0803508 − 0.0001520i	1.0803171 − 0.0000777i
35	1.1010513 − 0.0014642i	1.0803053 − 0.0000817i	1.0803177 − 0.0000355i
44	1.0948613 − 0.0019761i	1.0803153 − 0.0000246i	1.0803236 − 0.0000091i
54	1.0897742 − 0.0004540i	1.0803164 − 0.0000076i	1.0803167 − 0.0000029i
N	$d=4$	$d=8$	$d=9$
5	0.5104617 − 0.0896955i	–	–
9	1.2478041 − 0.0242760i	0.5884059 + 0.2937054i	–
14	1.0788934 − 0.0044051i	1.0816066 − 0.0127262i	1.0886293 − 0.0079029i
20	1.0804514 − 0.0003947i	1.0798093 − 0.0004621i	1.0795210 − 0.0008626i
27	1.0803508 − 0.0001072i	1.0803150 − 0.0001056i	1.0802968 − 0.0000844i
35	1.0803253 − 0.0000262i	1.0802855 − 0.0000483i	1.0803084 − 0.0000192i
44	1.0803130 − 0.0000063i	1.0803152 − 0.0000046i	1.0803166 − 0.0000064i
54	1.0803164 − 0.0000022i	1.0803169 − 0.0000018i	1.0803161 − 0.0000022i

d is the degree of the algebraic equation defining the approximant. N is the number of series coefficients used.

Table 2
Estimates $t_{c,N}$ and α_N of t_c and α for the series (27) obtained by differential approximants as N increases

N	$d=2$		$d=3$	
	$t_{c,N}$	α_N	$t_{c,N}$	α_N
5	0.0000000 + 1.7873652i	7.263 + 0.000i	–	–
9	1.0696264 − 0.0089512i	0.655 − 0.001i	0.0000000 − 2.0958632i	71.51 + 0.000i
14	1.0882841 − 0.0105829i	0.296 + 0.146i	1.1318756 − 0.0673574i	0.740 − 4.296i
20	1.0813245 + 0.0006382i	0.467 − 0.049i	1.0802258 + 0.0000849i	0.507 − 0.022i
27	1.0802842 + 0.0001397i	0.506 − 0.024i	1.0802843 + 0.0001229i	0.505 − 0.023i
35	1.0803058 + 0.0000549i	0.503 − 0.018i	1.0803213 + 0.0000307i	0.501 − 0.014i
44	1.0803152 + 0.0000183i	0.501 − 0.012i	1.0803141 + 0.0000095i	0.502 − 0.009i
54	1.0803151 − 0.0000066i	0.501 − 0.008i	1.0803160 + 0.0000049i	0.495 − 0.012i
N	$d=4$		$d=5$	
	$t_{c,N}$	α_N	$t_{c,N}$	α_N
5	–	–	–	–
9	0.0000000 + 7.9713856i	867.4 + 0.000i	–	–
14	1.1384340 + 0.0181276i	− 1.86 + 0.534i	0.0000000 − 4.7486422i	853.9 − 0.000i
20	1.0810490 + 0.0003045i	0.496 − 0.037i	1.0821553 + 0.0012680i	0.385 − 0.073i
27	1.0803447 − 0.0001092i	0.494 − 0.009i	1.0803680 + 0.0000326i	0.495 − 0.016i
35	1.0803025 + 0.0000251i	0.500 − 0.013i	1.0803102 + 0.0000242i	0.502 − 0.013i
44	1.0803145 + 0.0000128i	0.502 − 0.011i	1.0803161 + 0.0000063i	0.501 − 0.008i
54	1.0803160 − 0.0000037i	0.501 − 0.008i	1.0803170 + 0.0000032i	0.499 − 0.007i

The order of the differential equation defining the approximant is $d-1$. N is the number of series coefficients used.

Then, using the algebraic approximants of Drazin and Tourigny [5], the formulae (47) and (48) with $\lambda = t$ provide estimates for t_c, U_0 and U_1; these are listed in Table 3. The estimates appear to converge, albeit slowly, as N increases.

Table 3
Estimates $t_{c,N}$, $U_{0,N}$ and $U_{1,N}$ of t_c, U_0 and U_1 for the series (27), obtained by the method of Drazin and Tourigny [5] as N increases

d	N	$t_{c,N}$	$U_{0,N}$	$U_{1,N}$
2	5	1.0214806 + 0.0875912i	0.4306569 + 0.4656750i	0.7380787 + 0.0000000i
3	9	1.0977418 − 0.0195386i	0.3867584 + 0.6122710i	0.8628543 + 0.0554563i
4	14	1.0787921 − 0.0023551i	0.4298862 + 0.5645777i	0.7084770 + 0.0059807i
5	20	1.0802904 − 0.0005515i	0.4245018 + 0.5572135i	0.7186609 + 0.0481350i
6	27	1.0802553 − 0.0000889i	0.4250058 + 0.5544186i	0.7079524 + 0.0686477i
7	35	1.0803171 − 0.0000372i	0.4244774 + 0.5538710i	0.7134257 + 0.0770018i
8	44	1.0803170 − 0.0000062i	0.4245040 + 0.5534665i	0.7122967 + 0.0845764i
9	54	1.0803160 − 0.0000010i	0.4245355 + 0.5533479i	0.7110210 + 0.0885909i

d is the degree of the algebraic equation defining the approximant. N is the number of series coefficients used.

As a function of e, the vortex sheet has a singularity at $e = 0$ when $t = t_c$. As often when dealing with singularities, it is illuminating to think of the independent variable as *complex*. Cowley et al. [4] argued that, for every $t > 0$, there exist singularities of z in the complex e plane. They conjectured that there is a particular singularity at $e = e_s$, say, that moves along the imaginary axis as t increases until it reaches the origin at $t = t_c$. By using an approximation of the Birkhoff–Rott equation valid for $\mathrm{Im}\, e \gg 1$, Cowley et al. [4] obtained the formula

$$e_s \sim \mathrm{i} \ln\left(\frac{2}{at}\right) \quad \text{as } t \to 0+. \tag{29}$$

The series (7) can in fact be used to compute e_s directly. Indeed, let us write z as a Fourier series

$$z(e,t) = e + \sum_{n\in\mathbb{Z}} A_n(t) \exp(\mathrm{i}ne). \tag{30}$$

Using the fact that $A_0 = 0$ and that $A_{-n} = -A_n$ for $n \geqslant 1$, we obtain

$$z(e,t) = e + F(\exp(\mathrm{i}e)) - F(\exp(-\mathrm{i}e)), \tag{31}$$

where F is the power series

$$F(\xi) = \sum_{n=1}^{\infty} A_n(t)\xi^n. \tag{32}$$

If F has a singularity at $\xi = \xi_s(t)$ of the form

$$F(\xi) - F_0 \sim F_1(\xi_s - \xi)^{\gamma} \quad \text{as } \xi \to \xi_s, \tag{33}$$

then z has singularities at

$$e_s(t) = \pm[\arg \xi_s - \mathrm{i} \ln|\xi_s|]. \tag{34}$$

Furthermore, if we assume that there no other singularities nearer to the real axis, then the singularity ξ_s in the upper half plane determines the decay of A_n as $n \to \infty$. More precisely, as shown in Appendix A of [4],

$$A_n \sim C_\gamma n^{-\gamma-1} \exp(-n[\ln|\xi_s| + \mathrm{i}\arg\xi_s]) \quad \text{as } n \to \infty.$$

Now, the Fourier coefficient $A_n(t)$ is not known exactly, but the partial sums

$$A_n^{(K)} := \frac{1}{2\mathrm{i}} \sum_{k=n}^{K} Z_k^{(n)} t^k = A_n(t) + \mathrm{O}(t^{K+1}) \quad \text{as } t \to 0.$$

of its Taylor series can be used to approximate it. The results shown in Table 4 are obtained by using differential approximants for the series

$$\sum_{n=1}^{\infty} A_n^{(K)} \xi^n.$$

with $K = 60$. We believe them to be accurate to the number of decimals shown.

These results are in complete agreement with those of Cowley et al. [4]: A 3/2-type singularity moves from infinity along the imaginary axis and reaches the real axis at $t = t_c$ (see Fig. 2). The fact that $\gamma = 3/2$ implies that

$$A_n(t_c) = O(n^{-5/2}) \quad \text{as } n \to \infty,$$

as predicted by Moore [11]. However, it should be noted that, as $t \to t_c$, the coefficients $A_n^{(K)}$ become less accurate approximations of the true Fourier coefficients A_n, and so it is difficult to obtain good estimates of ξ_s and γ very near to the critical time. Nevertheless, our calculations suggest that the nature of the singularity does not change in that limit.

We end this section by remarking that we have performed similar calculations for a wide range of values of the parameter a. In particular, we have computed the critical time to a relative accuracy that

Table 4
The complex singularity (33) as a function of time

t_c/t	$\lvert\xi_s\rvert$	$\arg \xi_s$	γ	$\ln\lvert\xi_s\rvert/\ln(2/at)$
2	2.82079	0.00000	1.5000 + 0.0000i	0.7922
4	6.48217	0.00000	1.5000 + 0.0000i	0.9335
8	13.8645	0.00000	1.5000 + 0.0000i	0.9755
16	28.6632	0.00000	1.5000 + 0.0000i	0.9902
32	58.2782	0.00000	1.5000 + 0.0000i	0.9960
64	117.517	0.00000	1.5000 + 0.0000i	0.9983
128	235.999	0.00000	1.5000 + 0.0000i	0.9993
256	472.966	0.00000	1.5000 + 0.0000i	0.9997
512	946.901	0.00000	1.5000 + 0.0000i	0.9999
1024	1894.77	0.00000	1.5000 + 0.0000i	0.9999

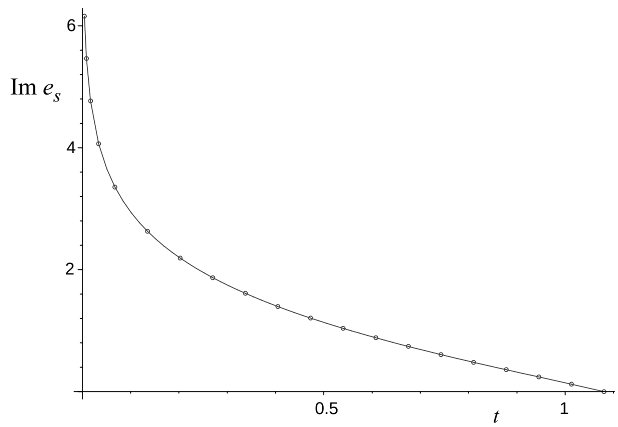

Fig. 2. The imaginary part of $e_s(t)$, the singularity of z in the complex e plane. Note that $\text{Im}\, e_s(t) \to 0$ as $t \to t_c = 1.08$.

is no less than 10^{-4} and have thus obtained the solid curve shown in Fig. 1. The results confirm that the formula (6) found by Moore is correct in the limit $a \to 0$. When $0 < a < 1$, the formula slightly underestimates the critical time, but still yields a remarkably useful approximation even when a is not so small.

Though the parameters t_c, U_0 and U_1 in the model (28) change with a, the exponent α itself appears to be independent of a. Furthermore, our calculations suggest that, regardless of the particular value of a, the singularity forms according to the same scenario as in the case $a = 1$: At $t = 0+$, a $3/2$-type singularity appears at infinity in the complex e plane and moves down along the imaginary axis until it reaches the origin at $t = t_c$.

5. Disturbances of infinite initial amplitude

Moore's analysis loses its validity as $a \to \infty$. Although this limit has little physical relevance, its study is of some mathematical interest and completes our understanding of the global dependence of the singularities upon the parameter a.

The results in Table 5 indicate that at_c approaches a limiting value as $a \to \infty$. To study this limit, we introduce the new variable

$$\tau = at,$$

so that (1) becomes

$$\frac{\partial \hat{z}^*}{\partial \tau}(e, \tau) = \frac{1}{4\pi \mathrm{i}} \mathrm{P.V.} \int_0^{2\pi} \left[\frac{1}{a} + \cos e'\right] \cot\left(\frac{\hat{z}(e, \tau) - \hat{z}(e', \tau)}{2}\right) \mathrm{d}e'. \tag{35}$$

$\hat{z}$ may be obtained in the form

$$\hat{z}(e, \tau) = e + \sum_{n=1}^{\infty} \tau^n \left(\sum_{k=0}^{n} \hat{z}_n^{(k)}(1/a) \sin(ke)\right), \tag{36}$$

where $\hat{z}_n^{(k)}$ is a polynomial of degree n. Of course, for $0 \neq a < \infty$, this expansion is equivalent to the MacLaurin series (7) in powers of t, in the sense that the finite partial sums carry precisely the same information. All we have done is to rearrange the series in a way that is computationally convenient when investigating the limit $a \to \infty$. Indeed, the Taylor coefficients of $\hat{z}$ may be computed by using recurrence relations analogous to those of Section 2.

Before launching into a detailed calculation of the dominant singularity of $\hat{z}$, it is useful to study the Fourier version of Eq. (35) in the manner of Moore. Let us write

$$\hat{z}(e, \tau) = e + \sum_{n \in \mathbb{Z}} \hat{A}_n(\tau) \exp(\mathrm{i}ne). \tag{37}$$

Table 5
at_c as a increases

a	2	4	8	16	32	64	128	256
at_c	1.345	1.557	1.702	1.788	1.836	1.861	1.874	1.881

Following in Moore's footsteps, we can then express Eq. (35) in the equivalent form

$$\frac{\mathrm{d}\hat{A}_n^*}{\mathrm{d}\tau} = \frac{1}{4}\delta_{n1} + \sum_{\ell=1}^{\infty}\left\{\frac{1}{a}\sum_{\substack{n_1+\cdots+n_\ell=n\\ n_1,\ldots,n_\ell\in\mathbb{Z}}} I(n_1,\ldots,n_\ell)\hat{A}_{n_1}\cdots\hat{A}_{n_\ell} + \frac{1}{2}\sum_{\substack{n_1+\cdots+n_\ell=n-1\\ n_1,\ldots n_\ell\in\mathbb{Z}}} J(n_1,\ldots,n_\ell)\hat{A}_{n_1}\cdots\hat{A}_{n_\ell} + \frac{1}{2}\sum_{\substack{n_1+\cdots+n_\ell=n+1\\ n_1,\ldots,n_\ell\in\mathbb{Z}}} K(n_1,\ldots,n_\ell)\hat{A}_{n_1}\cdots\hat{A}_{n_\ell}\right\} \tag{38}$$

for $n\in\mathbb{N}$, where $\hat{A}_0 = 0$, $\hat{A}_n = -\hat{A}_{-n}$ for $n \leqslant -1$ and δ is the familiar Kronecker delta. In this expression

$$I(n_1,\ldots,n_\ell) = \frac{1}{2\pi\mathrm{i}}\mathrm{P.V.}\int_{-\infty}^{\infty}\left(\prod_{m=1}^{\ell} 1-\exp(\mathrm{i}n_m u)\right)\frac{\mathrm{d}u}{u^{\ell+1}},$$

$$J(n_1,\ldots,n_\ell) = \frac{1}{2\pi\mathrm{i}}\mathrm{P.V.}\int_{-\infty}^{\infty}\exp(\mathrm{i}u)\left(\prod_{m=1}^{\ell} 1-\exp(\mathrm{i}n_m u)\right)\frac{\mathrm{d}u}{u^{\ell+1}}, \tag{39}$$

and

$$K(n_1,\ldots,n_\ell) = \frac{1}{2\pi\mathrm{i}}\mathrm{P.V.}\int_{-\infty}^{\infty}\exp(-\mathrm{i}u)\left(\prod_{m=1}^{\ell} 1-\exp(\mathrm{i}n_m u)\right)\frac{\mathrm{d}u}{u^{\ell+1}}.$$

This is an infinite system of differential equations for the Fourier coefficients $\hat{A}_n$ which must be solved subject to the initial condition

$$A_n(0) = 0 \quad \forall n\in\mathbb{Z}. \tag{40}$$

The nth Fourier coefficient is of the form

$$\hat{A}_n = \tau^{|n|}\hat{A}_{n0} + \tau^{|n|+2}\hat{A}_{n2} + \tau^{|n|+4}\hat{A}_{n4} + \cdots \tag{41}$$

We gain some insight into the properties of $\hat{z}$ for large a by substituting this expansion in (38). Setting $1/a = 0$ and retaining the terms of $\mathrm{O}(\tau^{|n|-1})$, we find

$$n\hat{A}_{n0}^* = \frac{1}{4}\delta_{n1} + \frac{1}{2}\sum_{\ell=1}^{n-1}\left\{\sum_{\substack{n_1+\cdots+n_\ell=n-1\\ n_1,\ldots,n_\ell\in\mathbb{N}}} J(n_1,\ldots,n_\ell)\hat{A}_{n_10}\cdots\hat{A}_{n_\ell 0}\right\} \tag{42}$$

for $n\in\mathbb{N}$, where $\hat{A}_{n0}(0) = 0$. Now, for positive values of $n_1,\ldots,n_\ell$, the integral (39) may be evaluated to yield

$$J(n_1,\ldots,n_\ell) = \frac{(-\mathrm{i})^\ell}{2}n_1,\ldots,n_\ell.$$

Using this, and setting

$$a_n = -\mathrm{i}n\hat{A}_{n0},$$

(42) becomes

$$a_{n+1}^* = \frac{\mathrm{i}}{4}\delta_{n0} + \frac{\mathrm{i}}{4}\sum_{\ell=1}^{n}\left\{\sum_{\substack{n_1+\cdots+n_\ell=n\\ n_1,\ldots,n_\ell\in\mathbb{N}}} a_{n_1}\cdots a_{n_\ell}\right\} \tag{43}$$

for $n = 0, 1, \ldots$ We introduce the generating function

$$G(x) = \sum_{n=1}^{\infty} a_n x^n, \quad x \in \mathbb{R}.$$

(3.4) is then equivalent to

$$G^* = \frac{\mathrm{i}x}{4}(1 + G + G^2 + \cdots) = \frac{\mathrm{i}x}{4}\frac{1}{1-G}.$$

This is readily solved to yield

$$G(x) = \frac{\mathrm{i}x}{4} + \frac{1}{2}\left(1 - \sqrt{1 - x^2/4}\right).$$

Hence, we obtain

$$\hat{A}_{n0} = \begin{cases} \frac{1}{4} & \text{if } n = 1, \\ \frac{\mathrm{i}}{2}\frac{1}{4^n}\frac{(n-2)!}{(n/2)!(n/2)!} & \text{if } n \text{ is even}, \\ 0 & \text{otherwise.} \end{cases}$$

By Stirling's formula

$$\hat{A}_{n0} \sim -\sqrt{2/\pi}\, n^{-5/2}\, 2^{-n} \quad \text{as } n \to \infty,\ n \text{ even},\ 1/a = 0.$$

This suggests that $z(e, t)$ has a singularity at $t = t_{\mathrm{c}}$ where

$$t_{\mathrm{c}} \approx \frac{2}{a} \quad \text{for } a \gg 1,$$

and that, at the singularity, the Fourier coefficients of z decay like $\mathrm{O}(n^{-5/2})$. In fact, however, the results in Table 5 indicate that at_{c} is slightly less than 2 for $a = \infty$. This small discrepancy with our analysis is not surprising because $\tau^{|n|}\hat{A}_{n0}$ cannot reasonably be expected to approximate $\hat{A}_n$ well when $\tau \approx 2$.

Let us now use approximants to obtain more reliable results. First, a calculation using the series summation technique of Drazin and Tourigny [5] yields (see Table 6)

$$\tau_c = 1.88712 \pm 10^{-5}. \tag{44}$$

If one assumes that

$$\frac{\partial \hat{z}}{\partial e}(0, \tau) - \hat{U}_0 \sim \hat{U}_1(\tau_c - \tau)^{\alpha} \quad \text{as } \tau \to \tau_c, \tag{45}$$

then differential approximants give, as in the case where a is finite,

$$\alpha \approx \frac{1}{2}.$$

Table 6
The approximations of τ_c, $\hat{U}_0$ and $\hat{U}_1$ by the method of Drazin and Tourigny for $2 \leqslant d \leqslant 11$

N	$\tau_{c,N}$	$\hat{U}_{0,N}$	$\hat{U}_{1,N}$
35	1.8871360 − 0.0000674i	0.3315759 + 0.7437507i	0.5056371 + 0.1807457i
44	1.8871187 − 0.0000354i	0.3317185 + 0.7435685i	0.5038936 + 0.1828748i
54	1.8871232 − 0.0000141i	0.3317443 + 0.7434008i	0.5031364 + 0.1853014i
65	1.8871224 − 0.0000031i	0.3317993 + 0.7432900i	0.5015523 + 0.1875465i
77	1.8871234 − 0.0000018i	0.3317956 + 0.7432677i	0.5014898 + 0.1881870i

Table 7
Estimates for the complex singularity as a function of τ

τ_c/τ	$\lvert\xi_s\rvert$	$\arg \xi_s$	γ	$\ln\lvert\xi_s\rvert/\ln(2/\tau)$
2	2.11067	0.00000	1.5000 + 0.0000i	0.9944
4	4.23688	0.00000	1.5000 + 0.0000i	0.9996
8	8.47761	0.00000	1.5000 + 0.0000i	0.9999
16	16.9566	0.00000	1.5000 + 0.0000i	1.0000
32	33.9139	0.00000	1.5000 + 0.0000i	1.0000
64	67.8281	0.00000	1.5000 + 0.0000i	1.0000
128	135.656	0.00000	1.5000 + 0.0000i	1.0000
256	271.313	0.00000	1.5000 + 0.0000i	1.0000
512	542.626	0.00000	1.5000 + 0.0000i	1.0000
1024	1085.25	0.00000	1.5000 + 0.0000i	1.0000

Estimates of $\hat{U}_0$ and $\hat{U}_1$ are listed in Table 6.

As in the previous section, we can study the evolution of the singularities of $\hat{z}$ in the complex e plane by considering the series

$$\hat{F}(\xi) = \sum_{n=1}^{\infty} \hat{A}_n(\tau)\xi^n.$$

We then find that $\hat{F}$ has a singularity at $\xi = \xi_s(t)$ of the form

$$\hat{F}(\xi) - \hat{F}_0 \sim \hat{F}_1\,(\xi_s - \xi)^{\gamma},$$

where γ and ξ_s are listed in Table 7. From these results, we deduce that, for $0 < \tau < \tau_c$, $\hat{z}$ has a 3/2-type singularity at $e = e_s(\tau)$ such that

$$e_s(\tau) \sim \mathrm{i}\ln\left(\frac{2}{\tau}\right) \quad \text{as } \tau \to 0.$$

This singularity moves down along the imaginary axis and reaches the origin at $\tau = \tau_c$.

6. Conclusion

In this paper, we have used power series to study the singularity that forms on a vortex sheet subject to the Kelvin–Helmholtz instability. By exploiting various generalisations of the Padé method, we have obtained accurate numerical approximations of the location and type of the singularity. One of our findings is that, if a is the amplitude of the initial perturbation in the vortex sheet strength, then the critical time t_c is given by

$$a\,t_c \sim 1.88712\ldots \quad \text{as } a \to \infty.$$

More generally, our results confirm the conjecture of Cowley et al. [4], according to which a singularity forms spontaneously in the complex e plane at $t = 0+$ and moves until it impinges on the real axis at a critical time t_c. This conjecture is based on an approximation valid for $\mathrm{Im}\, e \gg 1$. However, our calculations show that it is qualitatively correct even as $t \to t_c$. Remarkably, the nature of the singularity appears to be independent of t and a.

The results of this paper are significant in two different ways. First, they give new and more accurate results of the classic problem of Kelvin–Helmholtz instability. This we have described above. Secondly, they provide a basis for guidance about what method of summing power series should be chosen for many problems in fluid mechanics and similar subjects. We elaborate this guidance here.

The computing costs of finding the coefficients of a power series in a practical application are usually far higher than the costs of processing them by a summation method. So it behoves the user to exploit all the available information about the problem that gives rise to the series, without filtering or omitting data. This is one reason why our results are more accurate than those in Meiron et al. [10], which were obtained from a particular functional of the solution leading to a series in powers of t^2 rather than t.

Rational Padé approximation seems to have been used by almost all fluid dynamicists, in preference to (or perhaps ignorance of) nonlinear or differential approximants. A notable exception to this "rule" is Pelz and Gulak's use of a few nonlinear algebraic approximants to sum a series originating from an initial-value problem similar to the Taylor–Green problem [12]. This paper demonstrates the need not to restrict oneself to the Padé method.

It is a rule-of-thumb among the community who sum power series that if N coefficients of a series are used then each coefficient must be known to N decimal places though, of course, no such simple rule can be expected to be universally applicable. Ely and Baker [7] made use of interval arithmetic to combat problems associated with round-off errors. For the most part, we have performed the calculations in exact rational arithmetic, thus avoiding the danger that very small errors in the early coefficients might undermine the accuracy of the approximant method. It is wise to follow our example whenever possible, but the difficulty of computing the coefficients for some problems may require the use of multiprecision floating-point arithmetic. We have done this in one instance, when computing the singularities of the vortex sheet, viewed as a function of the Lagrangian marker variable e. The tracing of complex singularities of functions defined by their Fourier series by use of approximant methods is an exciting, apparently new, development. One can think of many other problems where this approach may be used; see for instance Sulem et al. [16].

Rapid convergence of summation, when it takes place, gives great confidence that the error is in fact small and that the method of summation chosen not only gives accurate numerical results, but also gives the asymptotic *form* of the singularity beyond reasonable doubt. Superexponential convergence gives a conviction of correctness which mere monotonic convergence does not. While some of the methods used in this study have been relatively successful when compared to those used previously, it is fair to say that we have no yet found an ideal method of summation for the Kelvin–Helmholtz problem. There is every incentive to continue the search for better methods, for though the detailed results obtained by them can never amount to a proof of the existence or form of the singularity, they can nevertheless guide the mathematician in forming plausible conjectures and perhaps suggest the method of proof.

Note added in proof

Professor Drazin passed away in January 2002 while this paper was under review.

Appendix A. Formulae for the singularity parameters of some Hermite–Pad approximants

Algebraic approximants. Let U_N be the approximant defined by Eq. (20). Then the singularities of U_N have the form (21). The location of the singularities in the complex λ plane can be found by solving simultaneously the equations

$$P_N^{(0)}(\lambda_{c,N})U_{0,N}^d + P_N^{(1)}(\lambda_{c,N})U_{0,N}^{d-1} + \cdots + P_N^{(d)}(\lambda_{c,N}) = 0, \tag{46}$$

$$dP_N^{(0)}(\lambda_{c,N})U_{0,N}^{d-1} + (d-1)P_N^{(1)}(\lambda_{c,N})U_{0,N}^{d-2} + \cdots + P_N^{(d-1)}(\lambda_{c,N}) = 0, \tag{47}$$

for the unknown pair $(\lambda_{c,N}, U_{0,N})$. The exponent α_N and the coefficient $U_{1,N}$ can then easily be found by using Newton's polygon algorithm. However, it is well known that, in the case of algebraic equations, the only singularities that are structurally stable are simple turning points. Hence, in practice, one almost invariably obtains $\alpha_N = 1/2$, and the coefficient $U_{1,N}$ is then given by the formula

$$U_{1,N}^2 = \frac{U_{0,N}^d DP_N^{(0)}(\lambda_{c,N}) + U_{0,N}^{d-1} DP_N^{(1)}(\lambda_{c,N}) + \cdots + DP_N^{(d)}(\lambda_{c,N})}{\binom{d}{2} U_{0,N}^{d-2} P_N^{(0)}(\lambda_{c,N}) + (d-1)2U_{0,N}^{d-3}P_N^{(1)}(\lambda_{c,N}) + \cdots + P_N^{(d-2)}(\lambda_{c,N})}, \tag{48}$$

where D denotes differentiation with respect to λ.

Differential approximants. Next, we consider the case where U_N is the approximant that solves the differential equation (22). Then, generically, the singularities $\lambda_{c,N}$ of U_N correspond to zeroes of the leading coefficient. Hence

$$P_N^{(0)}(\lambda_{c,N}) = 0. \tag{49}$$

If we assume a singularity of the form (21), then

$$\alpha_N = d - 2 - \frac{P_N^{(1)}(\lambda_{c,N})}{DP_N^{(0)}(\lambda_{c,N})}. \tag{50}$$

It has already been mentioned that the singularities of algebraic approximants are almost invariably of square-root type. The singularities of differential approximants are not so restricted. Hence, differential approximants should be preferred if the exponent α in the model (19) is the parameter of interest.

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